The Medial Feature Detector: Stable Regions from Image Boundaries
Supplementary Material

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Abstract

We give here supplementary material that can be helpful in the study of our work but could not fit into our submitted paper. In particular, we include the outline of our WMA and MAD algorithms, some additional comments on our partitioning scheme of section 5, especially in relation to watershed segmentation, as well as proofs of a number of our theoretical results. The remaining proofs are lengthier and will appear in a future publication.

Algorithm outlines

The algorithm outline of *Weighted Medial Axis* (WMA), presented in section 4, is given in algorithm 1. Parameter scale is used to prune the medial axis. We focus on removal of discretization effects only, not on simplification. A typical value is scale $= 2$ (pixels). In this case proposition 4.4 guarantees that the medial axis remains connected.

Similarly, the algorithm outline of *Medial Axis Decomposition* (MAD), presented in section 5, is given in algorithm 2.

Comments on partitioning

Propagation in our *medial axis decomposition* (MAD) scheme is equivalent to applying *watershed segmentation* to the negated distance map restricted to the medial axis (*i.e.* on $-\mathcal{A}(f)$) with peaks as markers.

However: (a) due to group marching, complexity is linear in k, where $k = |A(f)|$. (b) We ensure a single point per marker even in flat areas (*plateaus*), in which case this point is chosen at random; effectively, we build the connected components of the markers in parallel to propagation. (c) We construct graph $\mathcal{G}$, again in parallel. (d) What is not shown in the algorithm outline of MAD, is that given an edge $e = (u, v)$ generated at saddle point x, we *contract* e and identify u with v whenever $|h(u) - h(x)| \leq 1$ or $|h(v) - h(x)| \leq 1$. We thereby remove discretization effects along ridges while retaining true peaks.

Algorithm 1 Weighted Medial Axis

procedure MEDIAL(distance map h, source map s)
  initialize q, r; construct seed A_+ as in (5)
  for $x \in X$ **do** $r(x) \leftarrow 0$; label x as `far`
  for $x \in X$ **if** $x \succ x$ **then** label x as `done`
  for $x \in A_+$ **do** PROP(x)
  while $\neg\, q$.EMPTY( ) **do**
    $x \leftarrow q$.POP( ); label x as `done`
    for $y \diamond x$, $\neg\, y$ `done` **do** SCAN(x, y)
    if $r(x) \neq 0$ **for** $y \ast x$, y `far` **do** PROP(y)
  end while
  return residue r
end procedure

procedure PROP(point x)
  q.PUSH(x); label x as `near`
end procedure

procedure SCAN(point x, point y)
  $\rho \leftarrow \mathrm{res}(x, y)$
  if $s(x) = s(y) \vee \rho < \mathrm{scale}$ **return**
  if $\rho > r(y) \wedge y$ `far` **then** PROP(y)
  $r(x) \leftarrow \max(r(x), \rho)$; $r(y) \leftarrow \max(r(y), \rho)$
end procedure

As detailed in the paper, our image partitioning scheme is implemented by invoking EGM algorithm for a second time. Note however that due to proposition 5.1(c), we do not actually need the distance map output in this case; all we need is its *source map* output s. For this reason we use an even faster implementation of EGM where we discard all distance computations and all we do is *source backpropagation*.

Proofs

Proof of Lemma 3.1 (a) $\rightarrow$ (b). Using definitions (3), (2) and (1), it follows that if y is a source,

$$d(x, y) + f(y) \leq d(x, z) + f(z) \quad \forall z \in X \tag{S1}$$

Algorithm 2 Medial Axis Decomposition

procedure MAD(distance map h, medial axis A)
  initialize $q, \mathcal{G}$; construct $\hat{A}_+$
  for $x \in A$ **do** $\kappa(x) \leftarrow \emptyset$; label x as `far`
  for $x \in \hat{A}_+$ **do** PROP(x)
  while $\neg\, q$.EMPTY( ) **do**
    $x \leftarrow q$.POP( ); label x as `done`;
    for $y \ast x$, $y \in A$ **do** SCAN(x, y)
    if $\kappa(x) = \emptyset$ **then** $\kappa(x) \leftarrow \mathcal{G}$.VERTEX$(x)$
  end while
  return graph $\mathcal{G}$
end procedure

procedure PROP(point x)
  q.PUSH$(x, \lfloor -h(y) \rfloor)$; label x as `near`
end procedure

procedure SCAN(point x, point y)
  if y `far` **then** PROP(y)
  if $\kappa(y) = \emptyset$ **return**
  if $\kappa(x) = \emptyset$ **then** $\kappa(x) \leftarrow \kappa(y)$; **return**
  if $\kappa(x) \neq \kappa(y)$ **then** $\mathcal{G}$.EDGE$(\kappa(x), \kappa(y), h(x))$
end procedure

for some $x \in X$, or

$$f(y) \leq d(x, z) - d(x, y) + f(z) \quad \forall z \in X. \tag{S2}$$

Now, using the triangle inequality and the fact that $d(y, y) = 0$, we derive that

$$d(y, y) + f(y) \leq d(y, z) + f(z) \quad \forall z \in X, \tag{S3}$$

which, similarly to (S1), implies that $y \succcurlyeq y$.
(b) $\to$ (c). If $y \succcurlyeq y$, then by definition (2), $\mathcal{D}(f)(y) = f(y)$.
(c) $\to$ (d). If $\mathcal{D}(f)(y) = f(y)$, then by definition (2) $y \in \hat{S}(y)$, or $y \succcurlyeq y$. Suppose there is some other point $z \in S(y)$, then $z \in \hat{S}(y)$ or $z \succcurlyeq y \succcurlyeq y$. By definition (3) $z \notin S(y)$, a contradiction. Therefore $\hat{S}(y) = \{y\}$ implying that $S(y) = \{y\}$.
(d) $\to$ (e) and (d) $\to$ (a) are straightforward. $\square$

Proof of Lemma 3.2 Let

$$g(x) = \begin{cases} f(x), & x \in S(f) \\ +\infty, & \text{otherwise}. \end{cases} \tag{S4}$$

By definition (1), for all $x \in X$,

$$\mathcal{D}_d(g)(x) = \bigwedge_{y \in X} d(x, y) + g(y) \tag{S5}$$

$$= \bigwedge_{y \in S(f)} d(x, y) + f(y). \tag{S6}$$

On the other hand, it follows from definitions (1), (2), (3) that

$$\mathcal{D}_d(f)(x) = d(x, s(x)) + f(s(x)), \quad x \in X. \tag{S7}$$

But since $s(x) \in S(f)$, definition (1) for f gives

$$\mathcal{D}_d(f)(x) = \bigwedge_{y \in S(f)} d(x, y) + f(y), \quad x \in X. \tag{S8}$$

The above imply that $\mathcal{D}_d(f) = \mathcal{D}_d(g)$, where (by construction) g is uniquely determined by $f|_{S(f)}$, as claimed. $\square$

Proof of Lemma 4.1 Let $y \in S(f)$. By lemma 3.1, $S(y) = \{y\}$ hence $|S(y)| = 1$. Then y cannot be a medial point: $y \notin A(f)$. $\square$

Proof of Lemma 4.2 (a) Let $w \in S(x)$ be a source of x distinct from y. Considering x as an origin, define $\hat{y} = y - x$, $\hat{w} = w - x$, and $\hat{z} = z - x$ for $z \in L(x, y)$. Then $\hat{z} = \lambda \hat{y}$ for some $\lambda \in (0, 1)$. By lemma 4.1, vectors $\hat{y}$, $\hat{w}$, $\hat{z}$ are non-zero. Clearly,

$$\|\hat{y}\| = \|\hat{z}\| + \|\hat{y} - \hat{z}\|. \tag{S9}$$

On the other hand,

$$\|\hat{w}\| < \|\hat{z}\| + \|\hat{w} - \hat{z}\|. \tag{S10}$$

Here the triangle inequality is strict because otherwise we would have $\hat{w} = a\hat{y}$ with $a > 0$, meaning that either $w \succcurlyeq y$ (if $a > 1$) or $y \succcurlyeq w$ (if $a < 1$); but both cases contradict definition (3). Then

$$\|\hat{z} - \hat{y}\| - \|\hat{z} - \hat{w}\| < \|\hat{y}\| - \|\hat{w}\|. \tag{S11}$$

Adding $f(y) - f(w)$ to both sides, it follows that

$$d(z, y) + f(y) < d(z, w) + f(w), \tag{S12}$$

that is, w cannot be a source of z.

It remains to show that no other point $u \in X \setminus S(x)$ is a source of z. Suppose otherwise. Then $d(z, u) + f(u) \leq d(z, y) + f(y)$. Combining with (S9) and the triangle inequality,

$$d(x, u) + f(u) \leq d(x, y) + f(y), \tag{S13}$$

implying that $u \succcurlyeq y \succcurlyeq z$. Hence u is not a source of z, which is a contradiction.

Note: $L(x, y)$ is a special case of *shortest path* from source y to medial point x. It is a line segment because of the Euclidean space condition; in a general metric space, it would be replaced by a *geodesic*.

(b) With the topology induced by metric d, $N(x)$ contains an open ball centered at x, say $B_\epsilon(x)$. By (a) with

$0 < \lambda < \min(1, \epsilon/\|\hat{y}\|)$, there is $z = x + \lambda(y-x) \in N(x)$ with unique source $s(z) = y$.

(c) We know that $X \setminus A$ contains all points having a unique source. By (b), given $x \in A$, there is an open set U with $x \in U$ such that $U \cap (X \setminus A) \neq \emptyset$. Stated otherwise, x is a point of closure of $X \setminus A$. Since x is arbitrary, $A \subseteq \overline{X \setminus A}$. On the other hand, we know that $A \subseteq \overline{A}$. Therefore $A \subseteq \overline{X \setminus A} \cap \overline{A} = \partial A$.

Note: A is not necessarily closed, but if it is, $A = \partial A$. Of interest is the related concept of *cut locus*, which is defined exactly as the closure of A. □

Proof of Lemma 4.3 Given an input function f, lemma 4.1 suggests that the entire image support X is partitioned into the union of pairwise disjoint sets $S(f)$, $A(f)$ and $P(f) = I(f) \setminus A(f) = X \setminus (S(f) \cup A(f))$.

Given $z \in P(f)$ with unique source y, we can follow exactly the same line of reasoning as in the proof of lemma 4.2(a) to show that all points w on the shortest path $L(z, y)$ also have the same unique source, $S(w) = \{y\}$. Then, given *any* open line segment $L(x, y)$ such that $z \in L(x, y) \subset P(f)$, we still have $S(w) = \{y\}$ for all $w \in L(x, y)$; otherwise we would contradict our assumption that $S(z) = \{y\}$.

Given that X is bounded, such a line segment becomes *maximal* if x is either on the medial axis $A(f)$ or on the boundary ∂X w.r.t. domain $\mathbb{X}$; x cannot be on $S(f)$ because this would imply contradiction $x = y$. Let us extend the medial axis by

$$\dot{A}(f) = A(f) \cup \partial X. \tag{S14}$$

Conversely to lemma 4.2(a), we conclude that $P(f)$ consists of *all shortest paths* from sources y to points $x \in \dot{A}(f)$:

$$P(f) = \bigcup_{x \in \dot{A}(f)} \bigcup_{y \in S(x)} L(x, y). \tag{S15}$$

Again by lemma 4.2(a), all such sources are closure points of $P(f)$, therefore cannot be interior points of $S(f)$:

$$U(f) = \bigcup_{x \in \dot{A}(f)} \bigcup_{y \in S(x)} \{y\} \subseteq \partial S(f). \tag{S16}$$

Clearly, $U(f)$ is the set of sources of all points in $P(f) \cup A(f) = I(f)$. Now, by lemma 4.1, the medial axis $A(f)$ is uniquely determined by the set of sources of all points in the interior set $I(f)$, hence by $\partial S(f)$. □