Imperial College
MSc. Communications & Signal Processing
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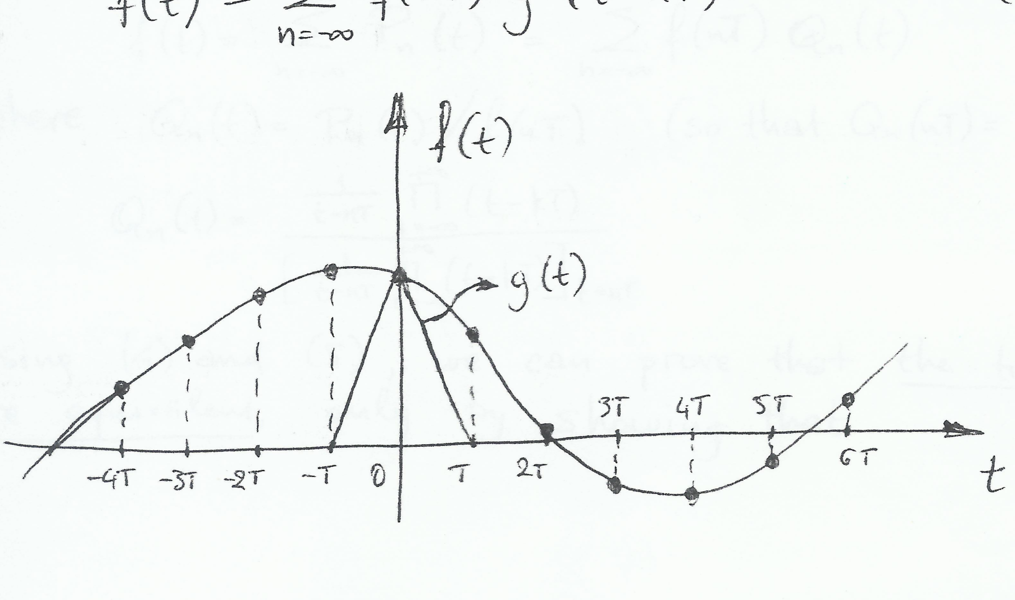
Reconstruction of continuous-time signals from sampled, discrete-time sequences with Lagrange interpolation polynomials

Reconstructing a continuous-time signal $f(t)$ from its sampled sequence $f(nT)$, where T is the sampling period, is usually done with the convolutional sum

$$f(t) = \sum_{n=-\infty}^{+\infty} f(nT)\, g\,(t-nT) \qquad (1)$$



where $g(t)$ is any function with the property

$$g(mT-nT)=\begin{cases}1, & \text{if } m=n\\ 0, & \text{if } m\neq n\end{cases} \qquad (2)$$

According to the <u>sampling theorem</u>, this can be achieved by passing $f(nT)$ through an ideal low-pass filter with transfer function

$$G(s)=\begin{cases}1, & |\omega-\omega_c|\leq 0\\ 0, & |\omega-\omega_c|>0\end{cases} \qquad (3)$$

where $\omega_c=\pi/T$. This is equivalent to

$$f(t)=\sum_{n=-\infty}^{+\infty} f(nT)\,\frac{\sin\omega_c(t-nT)}{\omega_c(t-nT)}=$$

$$=\sum_{n=-\infty}^{+\infty} f(nT)\,Sa\left[\pi(t-nT)/T\right] \qquad (4)$$

where $Sa(x)=\sin x/x$ is the sampling function.

Another way of doing so is by <u>Lagrange interpolation polynomials</u>. $f(t)$ is then given by the sum of an infinite number of polynomials $P_n(t)$ each of which has roots $t=kT$ for all the integer values of k, except $k=n$. For that value, $P_n(nT)=f(nT)$. This is expressed as

$$f(t)=\sum_{n=-\infty}^{+\infty} P_n(t)=\sum_{n=-\infty}^{+\infty} f(nT)\,Q_n(t) \qquad (5)$$

where $Q_n(t)=P_n(t)/f(nT)$ (so that $Q_n(nT)=1$), and

$$Q_n(t)=\frac{\frac{1}{t-nT}\prod_{k=-\infty}^{+\infty}(t-kT)}{\left[\frac{1}{t-nT}\prod_{k=-\infty}^{+\infty}(t-kT)\right]_{t=nT}} \qquad (6)$$

Using (4) and (5), we can prove that <u>the two methods are equivalent</u>, only by showing that

$$Q_n(t) = Sa\left[\pi(t-nT)/T\right] \quad \forall n \in \mathbb{Z} \qquad (7)$$

<u>Proof</u>.

a) For <u>n=0</u>, we have

$$Q_0(t) = \frac{\frac{1}{t}\prod_{k=-\infty}^{+\infty}(t-kT)}{\left[\frac{1}{t}\prod_{k=-\infty}^{+\infty}(t-kT)\right]_{t=0}} = \frac{\prod_{k=1}^{+\infty}(t-kT)(t+kT)}{\left[\prod_{k=1}^{+\infty}(t-kT)(t+kT)\right]_{t=0}} =$$

$$= \frac{\prod_{k=1}^{+\infty}(t^2-k^2T^2)}{\prod_{k=1}^{+\infty}(-k^2T^2)} = \prod_{k=1}^{+\infty}\left(1-\frac{t^2}{k^2T^2}\right) \qquad (8)$$

If we substitute $x \triangleq \frac{\pi t}{T}$, then

$$Q_0(t) = \prod_{k=1}^{+\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) \qquad (9)$$

But according to [1], this infinite product equals $Sa(x)$:

$$\prod_{k=1}^{+\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) = \frac{\sin x}{x} = Sa(x) \quad \forall x \in \mathbb{R} \qquad (10)$$

and therefore

$$Q_0(t) = Sa(\pi t/T) \qquad (11)$$

and (7) holds for $n=0$.

b) For <u>n≠0</u>, let $t' = t - nT$. Then

$$Q_n(t) = \frac{\frac{1}{t'}\prod_{k=-\infty}^{+\infty}(t'+nT-kT)}{\left[\frac{1}{t'}\prod_{k=-\infty}^{+\infty}(t'+nT-kT)\right]_{t'=0}} = \frac{\frac{1}{t'}\prod_{k'=-\infty}^{+\infty}(t'-k'T)}{\left[\frac{1}{t'}\prod_{k'=-\infty}^{+\infty}(t'-k'T)\right]_{t'=0}} = Q_0(t') \qquad (12)$$

where $k' = k - n$, and

$$Sa\left[\pi(t-nT)/T\right] = Sa\left[\pi t'/T\right] \qquad (13)$$

Thus eq. (7) becomes

$$Q_0(t') = Sa(\pi t'/T) \qquad (14)$$

which is identical to case (a) and therefore holds.

<u>Q.E.D.</u>

Reference

[1] Euler, L. "Introduction to Analysis of the Infinite", 1988, Springer-Verlag NY Inc., Book 1, pp. 116-128.