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CHAPTER IX

On Trinomial Factors.

143. The means by which linear factors of any polynomial may be found, we have seen above, is through the solution of an equation. If the polynomial is $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \cdots$ and a linear factor is of the form $p - qz$, then it is clear that whenever $p - qz$ is a factor of the function $\alpha + \beta z + \gamma z^2 + \cdots$, and when we substitute $\frac{p}{q}$ for z, then the factor $p - qz$ becomes zero and the proposed function vanishes. It follows that $p - qz$ is a factor or divisor of the polynomial $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \cdots$ whenever $\alpha + \beta \frac{p}{q} + \gamma \frac{p^2}{q^2} + \delta \frac{p^3}{q^3} + \epsilon \frac{p^4}{q^4} + \cdots = 0$. Conversely, if all the roots $\frac{p}{q}$ of this equation have been extracted, they will give all of the linear factors of the proposed polynomial $\alpha + \beta z + \gamma z^2 + \delta z^3 + \cdots$, that is $p - qz$. It is clear now that the number of these linear factors is determined by the greatest power of z.

144. From time to time it happens that complex linear factors are found only with difficulty. It is for this reason that I present in this chapter a special method by which the complex linear factors can frequently be found. Since complex linear factors are so paired that the product of two of them is real. We will find those complex factors if we study the quadratic factors of the form

$p - qz + rz^2$ which are real, but whose linear factors are complex. If the function $\alpha + \beta z + \gamma z^2 + \delta z^3 + \cdots$ has only real quadratic factors in this form of a trinomial $p - qz + rz^2$, then all of the linear factors will be complex.

145. A trinomial $p - qz + rz^2$ has linear factors which are complex if $4pr > q^2$, that is if $\frac{q}{2\sqrt{pr}} < 1$. Since the sine and cosine of angles are less than 1, a trinomial $p - qz + rz^2$ has complex linear factors if $\frac{q}{2\sqrt{pr}}$ is equal to the sine or cosine of some angle. Now let $\frac{q}{2\sqrt{pr}} = \cos \phi$ or $q = 2\sqrt{pr} \cos \phi$, and the trinomial $p - qz + rz^2$ has complex linear factors. Lest some irrationality cause problems, we assume the trinomial has the form $p^2 - 2pqz \cos \phi + q^2 z^2$, whose complex linear factors are $qz - p(\cos \phi + i \sin \phi)$ and $qz - p(\cos \phi - i \sin \phi)$. It is clear that if $\cos \phi = \pm 1$, then $\sin \phi = 0$ and both factors will be equal and real.

146. Given a polynomial $\alpha + \beta z + \gamma z^2 + \delta z^3 + \cdots$, the complex linear factors can be found if the values of p, q, and the arc ϕ are such that the trinomial $p^2 - 2pqz \cos \phi + q^2 z^2$ is a factor of the function. In this case, the complex linear factors will be $qz - p(\cos \phi + i \sin \phi)$ and $qz - p(\cos \phi - i \sin \phi)$. For this reason, the given function vanishes if we substitute either $z = \frac{p}{q}(\cos \phi + i \sin \phi)$ or $z = \frac{p}{q}(\cos \phi - i \sin \phi)$. When each of these substitutions is made, we obtain two equations which can be solved for both the fraction $\frac{p}{q}$ and the arc ϕ.

147. It might seem at first that these substitutions for z would cause difficulties, but when we use some of the results treated in the preceding chapter, things go rather expeditiously. We have seen that $(\cos\phi \pm i \sin\phi)^n = \cos n\phi \pm i \sin n\phi$, so that the following formulas are used when substituting for the powers of z. In the first factor,

$z = \frac{p}{q}(\cos\phi + i\sin\phi)$, $z^2 = \frac{p^2}{q^2}(\cos 2\phi + i\sin 2\phi)$,

$z^3 = \frac{p^3}{q^3}(\cos 3\phi + i\sin 3\phi)$, $z^4 = \frac{p^4}{q^4}(\cos 4\phi + i\sin 4\phi)$, etc.

In the second factor, $z = \frac{p}{q}(\cos\phi - i\sin\phi)$, $z^2 = \frac{p^2}{q^2}(\cos 2\phi - i\sin 2\phi)$,

$z^3 = \frac{p^3}{q^3}(\cos 3\phi - i\sin 3\phi)$, $z^4 = \frac{p^4}{q^4}(\cos 4\phi - i\sin 4\phi)$, etc. For the sake of brevity we let $\frac{p}{q} = r$ and then make the substitutions to obtain the two equations $0 = \alpha + \beta r\cos\phi + \gamma r^2\cos 2\phi + \delta r^3\cos 3\phi + \cdots$

$+ \beta r i\sin\phi + \gamma r^2 i\sin 2\phi + \delta r^3 i\sin 3\phi + \cdots$ and

$0 = \alpha + \beta r\cos\phi + \gamma r^2\cos 2\phi + \delta r^3\cos 3\phi + \cdots$

$- \beta r i\sin\phi - \gamma r^2 i\sin 2\phi - \delta r^3 i\sin 3\phi - \cdots$.

148. If these two equations are added and subtracted, and in the latter case also divided by $2i$ we obtain the two real equations

$0 = \alpha + \beta r\cos\phi + \gamma r^2\cos 2\phi + \delta r^3\cos 3\phi + \cdots$ and

$0 = \beta r\sin\phi + \gamma r^2\sin 2\phi + \delta r^3\sin 3\phi + \cdots$. In fact, given the polynomial $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \cdots$ we can immediately write down the two equations. In the first we put, for each power of z, $z^n = r^n\cos n\phi$ and in the second $z^n = r^n\sin n\phi$. Since $\sin 0\phi = 0$ and $\cos 0\phi = 1$, for z^0 in the first equation we put 1 and in the second we put 0. If now we can find the two unknown quantities r and ϕ from the two equations, then, since $r = \frac{p}{q}$, we will have the trinomial factor, $p^2 - 2pqz\cos\phi + q^2z^2$, of the given function and so also the two complex linear factors.

149. If the first equation is multiplied by $\cos m\phi$ and the second by $\sin m\phi$, then by addition and subtraction the following equations result.

$0 = \alpha\cos m\phi + \beta r\cos(m-1)\phi + \gamma r^2\cos(m-2)\phi$

$+ \delta r^3\cos(m-3)\phi + \cdots$ and

$0 = \alpha\cos m\phi + \beta r\cos(m+1)\phi + \gamma r^2\cos(m+2)\phi$

$+ \delta r^3\cos(m+3)\phi + \cdots$. Any two equations of this kind determine the unknowns r and ϕ. Since frequently there are several different solutions, we obtain several different trinomial factors, indeed we obtain all such factors in this way.

150. In order that the use of these rules may become clearer, we will investigate trinomial factors of certain functions which occur rather frequently. Once we have these results, they will be ready at hand for future use. Let the first such function be $a^n + z^n$; we will determine the trinomial factors of the form $p^2 - 2pqz\cos\phi + q^2z^2$. When we let $r = \frac{p}{q}$ we have the following two equations: $0 = a^n + r^n\cos n\phi$ and $0 = r^n\sin n\phi$. The second of these equations gives $\sin n\phi = 0$, so that $n\phi = (2k+1)\pi$, or $n\phi = 2k\pi$, where k is an integer. We will treat these two cases separately, since the cosines are different, being respectively $\cos(2k+1)\pi = -1$ and $\cos 2k\pi = 1$. It should be clear that the choice will be $n\phi = (2k+1)\pi$, since with $\cos n\phi = -1$, we have

$0 = a^n - r^n$. Since $r = a = \frac{p}{q}$, we have $p = a$, $q = 1$, and $\phi = \frac{(2k + 1)\pi}{n}$. It follows that a factor of $a^n + z^n$ will be $a^2 - 2az \cos \frac{(2k + 1)}{n}\pi + z^2$. Since any integer can be substituted for k, several factors of this form will be produced, but not an infinite number. This is because when $2k + 1$ becomes larger than n the factors begin to reoccur. This is because $\cos(2\pi \pm \phi) = \cos \phi$, but this will become clearer from examples. If n is an odd number, when $2k + 1 = n$, then there is a quadratic factor $a^2 + 2az + z^2$. From this it does not follow that $(a + z)^2$ is a factor of $a^n + z^n$, since from section 148 we see that only one equation results. It is clear that only $a + z$ is the divisor of $a^n + z^n$. This rule applies whether $\cos \phi$ is equal to +1 or -1.

EXAMPLE

We will develop a few cases so that we can see more clearly what the factors are. In these cases we distinguish between the odd and even values of n. If $n = 1$ then the function is $a + z$ and the factor is $a + z$. If $n = 2$, then the function is $a^2 + z^2$ and the factor is $a^2 + z^2$. If $n = 3$ then the function is $a^3 + z^3$ and the factors are $a^2 - 2az \cos \frac{1}{3}\pi + z^2$ and $a + z$. If $n = 4$ then the function is $a^4 + z^4$ and the factors are $a^2 - 2az \cos \frac{1}{4}\pi + z^2$ and $a^2 - 2az \cos \frac{3}{4}\pi + z^2$. If $n = 5$ then the function is $a^5 + z^5$ and the factors are $a^2 - 2az \cos \frac{1}{5}\pi + z^2$, $a^2 - 2az \cos \frac{3}{5}\pi + z^2$, and $a + z$. If $n = 6$ then the function is $a^6 + z^6$ and the factors are $a^2 - 2az \cos \frac{1}{6}\pi + z^2$, $a^2 - 2az \cos \frac{3}{6}\pi + z^2$, and $a^2 - 2az \cos \frac{5}{6}\pi + z^2$. From these examples it is clear that all of the factors have been obtained when for $2k + 1$ all odd numbers less than n are substituted. In those cases when a perfect square is produced, only its square root is a factor.

151. If the given function is $a^n - z^n$, then a trinomial factor is $p^2 - 2pqz \cos \phi + q^2r^2$. If we let $r = \frac{p}{q}$, then $0 = a^n - r^n \cos n\phi$ and $0 = r^n \sin n\phi$. Once again $\sin n\phi = 0$, and $n\phi = (2k + 1)\pi$ or $n\phi = 2k\pi$. In this case, however, we make the second choice, so that $\cos n = 1$, with $0 = a^n - r^n$, and $r = \frac{p}{q} = a$. It follows that $p = a$, $q = 1$, and $\phi = \frac{2k\pi}{n}$, so that the trinomial factor will be $a^2 - 2az \cos \frac{2k\pi}{n} + z^2$. In this formula we let $2k$ be equal to all even integers no larger than n to obtain all factors. Concerning factors which are perfect squares, we follow the rule given above. First we let $k = 0$ to obtain $a^2 - 2az + z^2$ from which we take the square root, $a - z$. Likewise, if n is even and $2k = n$, then we obtain $a^2 + 2az + z^2$ and $a + z$ is a divisor of $a^n - z^n$.

EXAMPLE

As in the previous example we distinguish between the odd and even values of n. If $n = 1$, then the function is $a - z$ and the factor is $a - z$. If $n = 2$ then the function is $a^2 - z^2$ and the factors are $a - z$ and $a + z$. If $n = 3$, then the function is $a^3 - z^3$ and the factors are $a - z$ and $a^2 - 2az \cos \frac{2}{3}\pi + z^2$. If $n = 4$ then the function is $a^4 - z^4$ and the factors are $a - z$, $a^2 - 2az \cos \frac{2}{4}\pi + z^2$, and $a + z$. If $n = 5$ then the function is

$a^5 - z^5$ and the factors are $a - z$, $a^2 - 2az\cos\frac{2}{5}\pi + z^2$, and $a^2 - 2az\cos\frac{4}{5}\pi + z^2$. If $n = 6$, then the function is $a^6 - z^6$ and the factors are $a - z$, $a^2 - 2az\cos\frac{2}{6}\pi + z^2$, $a^2 - 2az\cos\frac{4}{6}\pi + z^2$, and $a + z$.

152. These examples confirm what had been stated earlier, namely, that every polynomial, can be expressed as the product of real linear factors and real quadratic factors. We have seen that functions with the form $a^n + z^n$ with any degree, can be expressed as a product of real quadratic factors and real linear factors. We progress to more complicated functions such as $\alpha + \beta z^n + \gamma z^{2n}$. If this function has two factors of the form $\eta + \theta z^n$, then the factorization is clear from what we have just considered. We will show how to resolve such a function $\alpha + \beta z^n + \gamma z^{2n}$ into real linear or real quadratic factor in the case where there are not two real factors of the form $\eta + \theta z^n$.

153. We consider this function $a^{2n} - 2a^n z^n \cos g + z^{2n}$ which cannot be expressed as the product of two real factors of the form $\eta + \theta z^n$. If we suppose one of the real quadratic factors to be $p^2 + 2pqz\cos\phi + q^2z^2$, when we let $r = \frac{p}{q}$ we obtain the following two equations:

$0 = a^{2n} - 2a^n r^n \cos g \cos n\phi + r^{2n}\cos 2n\phi$ and

$0 = \ - 2a^n r^n \cos g \sin n\phi + r^{2n}\sin 2n\phi$. If instead of the first equation, we have from section 149, when $m = 2n$, $0 = a^{2n}\sin 2n\phi - 2a^n r^n \cos g \sin n\phi$. This equation with the second equation above give $r = a$. Then $\sin 2n\phi = 2\cos g \sin n\phi$. Since $\sin 2n\phi = 2\cos n\phi \sin n\phi$, it follows that $\cos n\phi = \cos g$. Since $\cos(2k\pi \pm g) = \cos g$, we have $n\phi = 2k\pi \pm g$ and

$\phi = \frac{2k\pi \pm g}{n}$. We now have the general quadratic factor of the proposed form $a^2 - 2az\cos\frac{2k\pi \pm g}{n} + z^2$, and all factors appear when we let $2k$ be all even integers no greater than n, as we shall see in the following.

EXAMPLE

We consider the cases in which n is 1, 2, 3, 4, etc. If the function is $a^2 - 2az\cos g + z^2$, then the factor is $a^2 - 2az\cos g + z^2$. If the function is $a^4 - 2az^2\cos g + z^4$, then the two factors are $a^2 - 2az\cos\frac{g}{2} + z^2$, and $a^2 - 2az\cos\frac{2\pi \pm g}{2} + z^2$, that is, $a^2 + 2az\cos\frac{g}{2} + z^2$. If the function is $a^6 - 2a^3z^3\cos g + z^6$, then the three factors are $a^3 - 2az\cos\frac{g}{3} + z^2$, $a^2 - 2az\cos\frac{2\pi - g}{3} + z^2$, and $a^2 - 2az\cos\frac{2\pi + g}{3} + z^2$. If the function is $a^8 - 2a^4z^4\cos g + z^8$, then the four factors are $a^2 - 2az\cos\frac{g}{4} + z^2$, $a^2 - 2az\cos\frac{2\pi - g}{4} + z^2$, $a^2 - 2az\cos\frac{2\pi + g}{4} + z^2$, and $a^2 - 2az\cos\frac{4\pi \pm g}{4} + z^2$, that is, $a^2 + 2az\cos\frac{g}{4} + z^2$. If the function is $a^{10} - 2a^5z^5\cos g + z^{10}$, then the five factors are $a^2 - 2az\cos\frac{g}{5} + z^2$, $a^2 - 2az\cos\frac{2\pi - g}{5} + z^2$, $a^2 - 2az\cos\frac{2\pi + g}{5} + z^2$, $a^2 - 2az\cos\frac{4\pi - g}{5} + z^2$, and $a^2 - 2az\cos\frac{4\pi + g}{5} + z^2$. Again it is confirmed in these examples that polynomials can be expressed as the product of real linear and real quadratic factors.

154. Now we can go further and consider a function of the form $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}$, which certainly has one factor of the form $\eta + \theta z^n$, and we have seen how to express this as a product of real linear and real quadratic factors. The other factor is of the form $\iota + \kappa z^n + \lambda z^{2n}$, which, according to the preceding section, can also be expressed as a product of real linear and real quadratic factors. Next we consider the function $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n}$. This always has two real factors of the form $\eta + \theta z^n + \iota z^{2n}$ and these likewise can be expressed as products of real linear and real quadratic factors. Then we consider the function $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \zeta z^{5n}$, which always has one factor of the form $\eta + \theta z^n$, while the other factor is of the form just considered. It follows that this function can be expressed as a product of real linear and real quadratic factors. If there were any doubt that every polynomial can be expressed as a product of real linear and real quadratic factors, then that doubt by this time should be almost completely dissipated.

155. We can extend this factorization also to infinite series. For example, we have seen that $1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots = e^x$. We have also seen that $e^x = (1 + x/j)^j$, where j is an infinitely large number. It becomes clear now that the series $1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots$ has an infinite number of linear factors, all of them equal, namely to $1 + \frac{x}{j}$. If we remove the first term from this series to obtain $\frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots = e^x - 1 = (1 + x/j)^j - 1$. When we compare this with the form in section 151, where we let $a = 1 + \frac{x}{j}$, $n = j$, and $z = 1$, each factor has the form $(1 + x/j)^2 - 2(1 + x/j)\cos\frac{2k\pi}{j} + 1$. When all even integers are substituted for $2k$ we obtain all of the factors. However, when $2k = 0$ we obtain the perfect square $\frac{x^2}{j^2}$ as a factor. For the reasons given before we take only the square root, $\frac{x}{j}$. It follows that x is a factor of the function $e^x - 1$, but that is already obvious. To find the other factors we have to note that the arc $\frac{2k}{j}\pi$ is infinitely small and according to section 134 we have $\cos\frac{2k}{j}\pi = 1 - 2\frac{k^2}{j^2}\pi^2$. The other terms in the series are neglected since j is infinitely large. It follows that each factor has the form $\frac{x^2}{j^2} + \frac{4k^2}{j^2}\pi^2 + \frac{4k^2}{j^3}\pi^2 x$ and $e^x - 1$ is divisible by $1 + \frac{x}{j} + \frac{x^2}{4k^2\pi^2}$. Therefore $e^x - 1 = x\left(1 + \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^3}{1\cdot 2\cdot 3\cdot 4} + \cdots\right)$ and except for the factor x, it has the infinite product of factors

$$\left(1 + \frac{x}{j} + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x}{j} + \frac{x^2}{16\pi^2}\right)\left(1 + \frac{x}{j} + \frac{x^2}{36\pi^2}\right)$$
$$\left(1 + \frac{x}{j} + \frac{x^2}{64\pi^2}\right)\cdots.$$

156. Since all of these factors contain a term which is infinitely small $\frac{x}{j}$, which, since it is in each factor, and through the multiplication of all the factors which are $\frac{1}{2}j$ in number, there is produced a term $\frac{x}{2}$, so $\frac{x}{j}$ cannot be omitted. In order to avoid this inconvenience we consider the expression

$e^x - e^{-x} = (1 + x/j)^j - (1 - x/j)^j$

$= 2\left(\frac{x}{1} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots\right)$, since

$e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} + \cdots$. We compare this with the expression in section 151, with $n = j$, $a = 1 + \frac{x}{j}$, and $z = 1 - \frac{x}{j}$. It follows that the factor of this series will be

$a^2 - 2az \cos \frac{2k\pi}{n} + z^2 = 2 + \frac{2x^2}{j^2} - 2\left(1 - \frac{x^2}{j^2}\right) \cos \frac{2k}{j}\pi$

$= \frac{4x^2}{j^2} + \frac{4k^2}{j^2}\pi^2 - \frac{4k^2\pi^2x^2}{j^4}$, since $\cos \frac{2k}{j}\pi = 1 - \frac{2k^2\pi^2}{j^2}$. The function $e^x - e^{-x}$, therefore, is divisible by $1 + \frac{x^2}{k^2}\pi^2 - \frac{x^2}{j^2}$, however, we omit the term $\frac{x^2}{j^2}$, since even when multiplied by j, it remains infinitely small. Further, when $k = 0$, the factor will be x. For these reasons, the factors can be given in the order in which they are calculated:

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\left(1 + \frac{x^2}{16\pi^2}\right)$$

$$\left(1 + \frac{x^2}{25\pi^2}\right)\cdots \quad = x\left(1 + \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots\right).$$

We have given each of the factors, multiplied by a constant of the same form so that when the factors are actually multiplied, the resulting first term will be x.

157. In the same way, $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$

$= \frac{(1 + x/j)^j + (1 - x/j)^j}{2}$. When this expression is compared to $a^n + z^n$, where we let $a = 1 + \frac{x}{j}$, $z = 1 - \frac{x}{j}$, and $n = j$, we obtain each factor as

$a^2 - 2az \cos \frac{2ak + 1}{n}\pi + z^2$

$= 2 + \frac{2x^2}{j^2} - 2(1 - x^2/j^2) \cos \frac{2k + 1}{n}\pi$. Since

$\cos \frac{2k + 1}{j}\pi = 1 - \frac{(2k + 1)^2}{2j^2}\pi^2$, the factor takes the form

$\frac{4x^2}{j^2} + \frac{(2k + 1)^2}{j^2}\pi^2$, where we have omitted a term whose denominator is j^4.

Since each factor of $1 + \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$ should have the form $1 + \alpha x^2$, we reduce the factor already found to the desired form when we divide by $\frac{(2k + 1)^2}{j^2}\pi^2$. We then have the factors in the proper form $1 + \frac{4x^2}{(2k + 1)^2\pi^2}$.

It follows from this that we can find the infinite product by substituting for $2k + 1$ successively all odd integers. Therefore we have

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots$$

$$= \left(\frac{1 + 4x^2}{\pi^2}\right)\left(\frac{1 + 4x^2}{9\pi^2}\right)\left(\frac{1 + 4x^2}{25\pi^2}\right)\left(\frac{1 + 4x^2}{49\pi^2}\right)\cdots .$$

158. If we let x be an imaginary number, then these exponential expressions can be represented by sines and cosines of a real arc. Let $x = zi$, then

$$\frac{e^{zi} - e^{-zi}}{2i} = \sin z \quad = z - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{z^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots$$

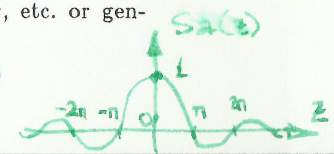
which has an expression as an infinite product:

$z(1 - z^2/\pi^2)(1 - z^2/4\pi^2)(1 - z^2/9\pi^2)(1 - z^2/16\pi^2)(1 - z^2/25\pi^2)\cdots$,

that is, we can write $\sin z = z(1 - z/\pi)(1 + z/\pi)(1 - z/2\pi)$

$(1 + z/2\pi)(1 - z/3\pi)(1 + z/3\pi)\cdots$. Whenever the arc z has a length such that any of the factors vanishes, that is when $z = 0, \pm\pi, \pm 2\pi$, etc. or gen-

$\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{(k\pi)^2}\right) = \frac{\sin z}{z} = Sa(z)$



erally when $z = \pm k\pi$, where k is any integer, then the sine of that arc must equal zero. But this is so obvious, that we might have found the factors from this fact. In like manner, since $\dfrac{e^{zi} + e^{-zi}}{2} = \cos z$ we also have $\cos z = (1 - 4z^2/\pi^2)(1 - 4z^2/9\pi^2)\,(1 - 4\pi^2/25\pi^2)(1 - 4z^2/49\pi^2)\cdots$, or when these factors are themselves factored, we obtain the expression

$\cos z = (1 - 2z/\pi)(1 + 2z/\pi)(1 - 2z/3\pi)(1 + 2z/3\pi)$

$(1 - 2z/5\pi)(1 + 2z/5\pi)\cdots$. From this it again becomes obvious that when $z = \pm\dfrac{2k+1}{2}\pi$, then $\cos z = 0$, which is clear from the nature of the circle.

159. From section 152 we can also find the factors of the expression

$$e^x - 2\cos g + e^{-x} = 2\left(1 - \cos g + \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots\right).$$

This expression can also be written as $(1 + x/j)^j - 2\cos g + (1 - x/j)^j$, in which we let $2n = j$, $a = 1 + \dfrac{x}{j}$, and $z = 1 - \dfrac{x}{j}$. It follows that each of the factors has the form

$$a^2 - 2az\cos\frac{2k\pi \pm g}{n} + z^2 = 2 + \frac{2z^2}{j^2} - 2(1 - x^2/j^2)\cos\frac{2(2k\pi \pm g)}{j}.$$

Since $\cos\dfrac{2(2k\pi \pm g)}{j} = 1 - \dfrac{2(2k\pi \pm g)^2}{j^2}$, the factor has the form $\dfrac{4x^2}{j^2} + \dfrac{4(2k\pi \pm g)^2}{j^2}$ or the form $1 + \dfrac{x^2}{(2k\pi \pm g)^2}$. If the expression is divided by $2(1 - \cos g)$, so that in the resulting infinite series the constant term is 1, then we have the following infinite product: $\dfrac{e^x - 2\cos g + e^{-x}}{2(1 - \cos g)}$

$$= \left(1 + \frac{x^2}{g^2}\right)\left(1 + \frac{x^2}{(2\pi - g)^2}\right)\left(1 + \frac{x^2}{(2\pi + g)^2}\right)\left(1 + \frac{x^2}{(4\pi - g)^2}\right)$$

$$\left(1 + \frac{x^2}{(4\pi + g)^2}\right)\left(1 + \frac{x^2}{(6\pi - g)^2}\right)\left(1 + \frac{x^2}{(6\pi + g)^2}\right)\cdots.$$

Furthermore, if we substitute zi for x, then

$$\frac{\cos z - \cos g}{1 - \cos g} = \left(1 - \frac{z}{g}\right)\left(1 + \frac{z}{g}\right)\left(1 - \frac{z}{2\pi - g}\right)\left(1 + \frac{z}{2\pi - g}\right)$$

$$\left(1 - \frac{z}{2\pi + g}\right)\left(1 + \frac{z}{2\pi + g}\right)\left(1 - \frac{z}{4\pi - g}\right)\left(1 + \frac{z}{4\pi - g}\right)\cdots$$

$$= 1 - \frac{z^2}{1\cdot 2(1 - \cos g)} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4(1 - \cos g)}$$

$$- \frac{z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(1 - \cos g)} + \cdots.$$

Now we have an infinite product expression for this infinite series.

160. It would be convenient to be able to find an infinite product expression for the function $e^{b+x} \pm e^{c-x}$. When we transform it into the form $\left(1 + \dfrac{b+x}{j}\right)^j \pm \left(1 + \dfrac{c-x}{j}\right)^j$ we can compare it with $a^j \pm z^j$, which has a factor $a^2 - 2az\cos\dfrac{m\pi}{j} + z^2$ where m is odd when the sign is positive and m is even when the sign is negative. Since j is infinitely large, $\cos\dfrac{m\pi}{j} = 1 - \dfrac{m^2\pi^2}{2j^2}$. Then the general factor has the form $(a - z)^2 + \dfrac{m^2\pi^2}{j^2}az$. In the present case we have $a = 1 + \dfrac{b+x}{j}$ and $z = 1 + \dfrac{c-x}{j}$, so that $(a - z)^2 = \dfrac{(b - c + 2x)^2}{j^2}$ and $az = 1 + \dfrac{b+c}{j} + \dfrac{bc + (c-b)x - x^2}{j^2}$. When these substitutions have been made and the result multiplied by j^2, we obtain

formation for these factors is sufficiently simple and uniform. Furthermore, from the multiplication of these expressions, there arise the expressions found in the previous section.

$$\frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}$$

$$\frac{\sin(\pi x)}{\pi x} = \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k+1)!} x^{2k} = f(x) = \sum_{k=0}^{\infty} c(k) x^{2k}$$

$$c(k) = \frac{(-1)^k \pi^{2k}}{(2k+1)!} \rightarrow c(k) = \frac{-\pi^2}{2k(2k+1)} c(k-1)$$

$$f(x) = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \Rightarrow$$

$$c(0) = 1.$$

$$c(1) = -\sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{\pi^2}{6}$$

$$c(2) = \frac{1}{2} \sum_{k=1}^{\infty} \left(-\frac{1}{k^2}\right)\left(c(1) + \frac{1}{k^2}\right) = \frac{1}{2}\left[c(1)^2 - \sum_{k=1}^{\infty} \frac{1}{k^4}\right].$$

$$c(3) = \frac{1}{3} \sum_{k=1}^{\infty} \left(-\frac{1}{k^2}\right)\left(c(2) + \frac{1}{k^2}\left(c(1) + \frac{1}{k^2}\right)\right) =$$

$$= \frac{1}{3}\left[c(1)c(2) - c(1)\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{1}{k^6}\right]$$

$$c(4) = \frac{1}{4} \sum_{k=1}^{\infty} \left(-\frac{1}{k^2}\right)\cdot\left(c(3) + \frac{1}{k^2}\left(c(2) + \frac{1}{k^2}\left(c(1) + \frac{1}{k^2}\right)\right)\right) =$$

$$= \frac{1}{4}\left[c(1)c(3) - c(2)\sum_{k=1}^{\infty} \frac{1}{k^4} - c(1)\sum_{k=1}^{\infty} \frac{1}{k^6} - \sum_{k=1}^{\infty} \frac{1}{k^8}\right].$$

$$c_n \triangleq c(n), \quad d_n \triangleq -\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \Longrightarrow$$

$$c_0 = 1$$

$$c_1 = d_1 = -\frac{\pi^2}{6}$$

$$c_2 = \frac{1}{2}[c_1 d_1 + d_2]$$

$$c_3 = \frac{1}{3}[c_2 d_1 + c_1 d_2 + d_3]$$

$$c_4 = \frac{1}{4}[c_3 d_1 + c_2 d_2 + c_1 d_3 + d_4]$$

$$c_5 = \frac{1}{5}[c_4 d_1 + c_3 d_2 + c_2 d_3 + c_1 d_4 + d_5]$$

$\vdots$

CHAPTER X

On the Use of the Discovered Factors to Sum Infinite Series.

165. If $1 + Az + Bz^2 + Cz^3 + Dz^4 + \cdots$ $= (1 + \alpha z)(1 + \beta z)(1 + \gamma z)(1 + \delta z) \cdots$, then these factors, whether they be finite or infinite in number, must produce the expression $1 + Az + Bz^2 + Cz^3 + Dz^4 + \cdots$, when they are actually multiplied. It follows then that the coefficient A is equal to the sum $\alpha + \beta + \gamma + \delta + \epsilon + \cdots$. The coefficient B is equal to the sum of the products taken two at a time. Hence $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \cdots$. Also the coefficient C is equal to the sum of products taken three at a time, namely $C = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta + \cdots$. We also have D as the sum of products taken four at a time, and E is the sum of products taken five at a time, etc. All of this is clear from ordinary algebra.

166. Since the sum $\alpha + \beta + \gamma + \delta + \cdots$ is given along with the sum of products taken two at a time, we can find the sum of the squares $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \cdots$, since this is equal to the square of the sum diminished by two times the sum of the products taken two at a time. In a similar way the sums of the cubes, biquadratics, and higher powers can be found. If we let $P = \alpha + \beta + \gamma + \delta + \epsilon + \cdots$

$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \cdots$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \cdots$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 + \cdots$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 + \cdots$$
$$V = \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 + \cdots .$$

Then P, Q, R, S, T, V, etc. can be found in the following way from A, B, C, D, etc. $P = A$, $Q = AP - 2B$, $R = AQ - BP + 3C$, $S = AR - BQ + CP - 4D$, $T = AS - BR + CQ - DP + 5E$, $V = AT - BS + CR - DQ + EP - 6F$, etc. The truth of these formulas is intuitively clear, but a rigorous proof will be given in the differential calculus.

167. Since we found above, in section 156, that

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{x^2}{1\cdot2\cdot3} + \frac{x^4}{1\cdot2\cdot3\cdot4\cdot5} + \frac{x^6}{1\cdot2\cdots7} + \cdots\right)$$
$$= x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\left(1 + \frac{x^2}{16\pi^2}\right)\left(1 + \frac{x^2}{25\pi^2}\right)\cdots,$$

it follows that

$$1 + \frac{x^2}{1\cdot2\cdot3} + \frac{x^4}{1\cdot2\cdot3\cdot4\cdot5} + \frac{x^6}{1\cdot2\cdots7} + \cdots$$
$$= \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{4\pi^2}\right)\left(1 + \frac{x^2}{9\pi^2}\right)\left(1 + \frac{x^2}{16\pi^2}\right)\cdots .$$

If we let $x^2 = \pi^2 z$,

$$1 + \frac{\pi^2}{1\cdot2\cdot3}z + \frac{\pi^4}{1\cdot2\cdot3\cdot4\cdot5}z^2 + \frac{\pi^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}z^3 + \cdots$$
$$= (1 + z)(1 + z/4)(1 + z/9)(1 + z/16)(1 + z/25)\cdots .$$

We use the rules stated above where $A = \frac{\pi^2}{6}$, $B = \frac{\pi^4}{120}$, $C = \frac{\pi^6}{5040}$, $D = \frac{\pi^8}{362880}$, etc., and we also have

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots,$$

$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \cdots,$$
$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \cdots,$$
$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \cdots,$$
$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \cdots .$$

From the values of A, B, C, D, etc. we see that $P = \frac{\pi^2}{6}$, $Q = \frac{\pi^4}{90}$, $R = \frac{\pi^6}{945}$, $S = \frac{\pi^8}{9450}$, $T = \frac{\pi^{10}}{93555}$, etc.

168. It is clear that any infinite series of the form $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$, provided n is an even integer, can be expressed in terms of π, since it always has a sum equal to a fractional part of a power of π. In order that the values of these sums can be seen even more clearly, we set down in a convenient form some more sums of these series.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{2^0}{1\cdot2\cdot3}\,\frac{1}{1}\pi^2$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4}\cdots = \frac{2^2}{1\cdot2\cdot3\cdot4\cdot5}\,\frac{1}{3}\pi^4$$
$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots = \frac{2^4}{1\cdot2\cdots7}\,\frac{1}{3}\pi^6$$
$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \cdots = \frac{2^6}{1\cdot2\cdot3\cdots9}\,\frac{3}{5}\pi^8$$
$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \cdots = \frac{2^8}{1\cdot2\cdot3\cdots11}\,\frac{5}{3}\pi^{10}$$
$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \cdots = \frac{2^{10}}{1\cdot2\cdot3\cdots13}\,\frac{691}{105}\pi^{12}$$

$$1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} + \frac{1}{5^{14}} + \cdots = \frac{2^{12}}{1\cdot 2\cdot 3 \cdots 15}\,\frac{35}{1}\pi^{14}$$

$$1 + \frac{1}{2^{16}} + \frac{1}{3^{16}} + \frac{1}{4^{16}} + \frac{1}{5^{16}} + \cdots = \frac{2^{14}}{1\cdot 2\cdot 3 \cdots 17}\,\frac{3617}{15}\pi^{16}$$

$$1 + \frac{1}{2^{18}} + \frac{1}{3^{18}} = \frac{1}{4^{18}} + \frac{1}{5^{18}} + \cdots = \frac{2^{16}}{1\cdot 2\cdot 3 \cdots 19}\,\frac{43867}{21}\pi^{18}$$

$$1 + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \frac{1}{4^{20}} + \frac{1}{5^{20}} + \cdots = \frac{2^{18}}{1\cdot 2\cdot 3 \cdots 21}\,\frac{1222277}{55}\pi^{20}$$

$$1 + \frac{1}{2^{22}} + \frac{1}{3^{22}} + \frac{1}{4^{22}} + \frac{1}{5^{22}} + \cdots = \frac{2^{20}}{1\cdot 2\cdot 3 \cdots 23}\,\frac{854513}{3}\pi^{22}$$

$$1 + \frac{1}{2^{24}} + \frac{1}{3^{24}} + \frac{1}{4^{24}} + \frac{1}{5^{24}} + \cdots = \frac{2^{22}}{1\cdot 2\cdot 3 \cdots 25}\,\frac{1181820455}{273}\pi^{24}$$

$$1 + \frac{1}{2^{26}} + \frac{1}{3^{26}} + \frac{1}{4^{26}} + \frac{1}{5^{26}} + \cdots = \frac{2^{24}}{1\cdot 2\cdot 3 \cdots 27}\,\frac{76977927}{1}\pi^{26}.$$

We could continue with more of these, but we have gone far enough to see a sequence which at first seems quite irregular, $1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, \frac{35}{1}, \ldots$, but it is of extraordinary usefulness in several places.

169. We now treat in the same manner the equation found in section 157. There we saw that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots$$

$$= \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{9\pi^2}\right)\left(1 + \frac{4x^2}{25\pi^2}\right)\left(1 + \frac{4x^2}{49\pi^2}\right)\cdots.$$ We let $x^2 = \frac{\pi^2 z}{4}$,

then $1 + \frac{\pi^2}{1\cdot 2\cdot 4}z + \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 4^2}z^2 + \frac{\pi^6}{1\cdot 2 \cdots 6\cdot 4^3}z^3 + \cdots$

$= (1 + z)(1 + z/9)(1 + z/25)(1 + z/49)\cdots$. We now use the formulas, where $A = \frac{\pi^2}{1\cdot 2\cdot 4}$, $B = \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 4^2}$, $C = \frac{\pi^6}{1\cdot 2\cdot 3 \cdots 6\cdot 4^3}$, etc., and

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots$$

$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \cdots$$

$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \cdots$$

$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \cdots.$$

It follows that $P = \frac{1}{1}\frac{\pi^2}{2^3}$, $Q = \frac{2}{1\cdot 2\cdot 3}\frac{\pi^4}{2^5}$, $R = \frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}\frac{\pi^6}{2^7}$,

$S = \frac{272}{1\cdot 2\cdot 3 \cdots 7}\frac{\pi^8}{2^9}$, $T = \frac{7936}{1\cdot 2\cdot 3 \cdots 9}\frac{\pi^{10}}{2^{11}}$, $V = \frac{353792}{1\cdot 2\cdot 3 \cdots 11}\frac{\pi^{12}}{2^{13}}$,

$W = \frac{22368256}{1\cdot 2\cdot 3 \cdots 13}\frac{\pi^{14}}{2^{15}}$.

170. The same sums of powers of odd numbers can be found from the preceding sums in which all numbers occur. If we let $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \cdots$ and multiply both sides by $\frac{1}{2^n}$, we obtain $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots$. This series contains only even numbers, which, when subtracted from the previous series, leaves the series with only odd numbers. Hence,

$M - \frac{M}{2^n} = \frac{2^n - 1}{2^n}M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \cdots$. If 2 times the series $\frac{M}{2^n}$ is subtracted from M an alternating series is produced:

$M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}}M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \cdots$. In this way we can sum the series